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Finite-Dimensional Quantum Mechanics

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The basic properties of nonrelativistic finite-dimensional quantum mechanics are presented. A discrete quantum mechanics is developed. Second quantization, the symmetric and antisymmetric Fock spaces are also discussed.

1. INTRODUCTION

There are three important reasons for studying finite-dimensional quantum mechanics. First, finite-dimensional quantum mechanics approximates ordinary quantum mechanics and the approximation gets better as the dimension increases. In this way, numerical methods can be utilized to an arbitrarily high precision. Second, there appear to be good reasons to believe in the existence of a fundamental unit of length (Atkinson and Halpern, 1967; Gudder, 1968; Heisenberg, 1930). If this turns out to be true, then the resulting quantum mechanics will be finite dimensional in the bounded case, and discrete in the unbounded case. Third, finite-dimensional quantum mechanics may lead to a deeper understanding of conventional quantum mechanics. As we shall see, the Hilbert space $L^2(\mathbb{R}^3)$ of a single nonrelativistic particle p is the second quantization of the finite-dimensional space $\mathbb{C}^3$. The particle p can then be described in terms of more elementary "particles" in $\mathbb{C}^3$. The quantum mechanics on $L^2(\mathbb{R}^3)$ can therefore be thought of as a finite-dimensional quantum field theory, and its study may shed light on the conventional quantum field theory.

In this paper we present the basic properties of nonrelativistic finite-dimensional quantum mechanics. We expect to consider relativistic theory and deeper aspects of finite-dimensional quantum mechanics, and quantum field theory in later works.

2. QUANTUM MECHANICS ON $\mathbb{C}^r$

In the sequel, $V=\mathbb{C}^r$ and $e_1,\ldots,e_r$ is the standard basis $e_j(k)=\delta_{jk}$ on V. The projection operator Q_j onto e_j is called the j**th location observable.** For any $f\in V$, we have $(Q_jf)(k)=\delta_{jk}f(j)$ and $Q_jf=f(j)e_j$. Any real function of the Q_j's is called a **location observable** and every location observable has the form $A=\Sigma g(j)Q_j=g\cdot\mathbf{Q}$ where $g\in\mathbb{R}^r$. For any $f\in V$, we have $(Af)(j)=g(j)f(j)$.

Define the **finite Fourier transform** (Auslander and Tolimieri, 1979) F: $V\to V$ as

$$(Ff)(j)=r^{-1/2}\sum_{k=1}^{r} f(k)e^{2\pi ijk/r}$$

Then F is unitary and

$$(F^*f)(j)=r^{-1/2}\sum_{k=1}^{r} f(k)e^{-2\pi ijk/r}$$

The matrix elements of F are $F_{jk}=\langle Fe_k,e_j\rangle=r^{-1/2}e^{2\pi ijk/r}$. The operator $P_j=F^*Q_jF$ is a projection onto the vector $f_j=F^*e_j$ and is called the j**th motion observable**. Any function $B=\Sigma g(j)P_j=g\cdot\mathbf{P}$ of the P_j's is called a **motion observable.**

The matrix elements of P_n are $(P_n)_{jk}=\langle P_ne_k,e_j\rangle=r^{-1}e^{2\pi in(k-j)/r}$.

Let Q be the location observable defined by $(Qf)(j)=jf(j)$, $Q=\Sigma jQ_j$. Thus, Q represents a location observation for a particle which can be located at one of the points $1,2,\ldots,r$. The eigenvectors of Q are e_j with corresponding eigenvalues j, $j=1,\ldots,r$. Let $P=F^*QF=\Sigma jP_j$ be the motion observable corresponding to Q. The eigenvectors of P are $f_j=F^*e_j$, with corresponding eigenvalues j, $j=1,\ldots,r$.

The motion observable P is the infinitesimal generator of a one-parameter unitary group $V(a)=e^{iaP}=\Sigma e^{iaj}P_j$. Letting $a=2\pi m/r$, $m\in N$, we have

$$\left[e^{i(2\pi m/r)P}\right]_{jk}=r^{-1}\sum_{n=1}^{r} e^{i2\pi mn/r}e^{2\pi in(k-j)/r}$$

$$=r^{-1}\sum_{n=1}^{r} e^{2\pi in(k-j+m)/r}=\sum_{n=1}^{r}(P_n)_{j,k+m}=\delta_{j,k+m}$$

(mod r) in m. Hence,

$$\left[e^{i(2\pi m/r)P}e_k\right]_j=\langle e^{i(2\pi m/r)P}e_k,e_j\rangle=\delta_{j,k+m}$$

and $e^{i(2\pi m/r)P}e_k = e_{k+m}$ (mod r). This is why P is called a motion observable; it generates a dynamical group $V(a)$ which moves a particle from location k to location $k+m$ (mod r) in time $2\pi m/r$. It follows that $[V(2\pi m/r)f](k) = f(k-m)$, (mod r), for all $f \in V$. One can also observe the motion property of P by viewing the dynamics $V(a)$ in the Heisenberg picture. A simple computation shows that $V(-2\pi m/r)QV(2\pi m/r) = Q + mI$ (mod r).

Since P and Q are bounded observables, the Heisenberg form of the commutation relation cannot hold. In order to derive $[Q, P]$ notice that

$$P_{jk} = \sum_{n=1}^{r} n(P_n)_{jk} = r^{-1} \sum_{n=1}^{r} ne^{2\pi in(k-j)/r}$$

Since

$$(QP)_{jk} = \sum_{n} Q_{jn} P_{nk} = \sum_{n} n\delta_{jn} P_{nk} = jP_{jk}$$

and similarly $(PQ)_{jk} = kP_{jk}$ we have

$$[Q, P]_{jk} = (j-k)P_{jk} = (j-k)r^{-1} \sum_{n=1}^{r} ne^{2\pi in(k-j)/r}$$

A discrete version of the Weyl form of the commutation relations does hold. In fact, if $s \in N$, and $t \in \mathbb{R}$, we have

$$e^{i(2\pi t/r)Q} e^{i(2\pi s/r)P} = e^{i2\pi ts/r} e^{i(2\pi s/r)P} e^{i(2\pi t/r)Q}$$

This follows from

$$\begin{aligned} e^{i(2\pi t/r)Q} e^{i(2\pi s/r)P} e_n &= e^{i(2\pi t/r)Q} e_{n+s} = e^{i[2\pi t(n+s)/r]} e_{n+s} \\ &= e^{i2\pi ts/r} e^{i2\pi tn/r} e_{n+s} = e^{i2\pi ts/r} e^{i(2\pi s/r)P} e^{i2\pi tn/r} e_n \\ &= e^{i2\pi ts/r} e^{i(2\pi s/r)P} e^{i(2\pi t/r)Q} e_n \end{aligned}$$

It should be mentioned that Q need not be an actual position observable—it could be a spin observable or any other observable with a finite number of values. Also, although the usual Heisenberg uncertainty principle does not hold in finite dimensions, it does hold in the following weaker sense. Suppose the system is in a movement state $f_j = F^* e_j$. Then P has an exact value j and dispersion zero. The probability that Q has value k

becomes

$$\langle Q_k f_j, f_j\rangle = |\langle e_k, f_j\rangle|^2 = |\langle Fe_k, e_j\rangle|^2 = |F_{jk}|^2 = r^{-1}$$

Since this holds for all $k=1,\ldots,r$, Q has uniform distribution. Thus, Q is completely undetermined in this state and has maximal dispersion. Of course, the analogous result also holds for location states.

The above theory converges to the usual one-degree of freedom quantum mechanics in the following sense. Let $H=L^2[0,1]$ and let q and p be the usual position and momentum operators. Then there exist subspaces H_r of dimension r such that $H_r \to H$ in that for every $f \in H$ there exists a sequence $f_r \in H_r$ such that $f_r \to f$ and for every such sequence f_r, the sequence of location observables q_r satisfies $q_r f_r \to qf$ and for every $f \in D(p)$, the sequence of motion observables p_r satisfies $p_r f_r \to pf$. To see this, let $\mathcal{P}_r = \{0, r^{-1}, 2r^{-1}, \ldots, 1\}$ be a partition of $[0,1]$ and let H_r be the set of step functions which are constant on the intervals of $\mathcal{P}_r$. Then H_r is an r-dimesnional subspace of H and H_r is isomorphic to $\mathbb{C}^r$. Now define q_r: $H_r \to H_r$ by $(q_r f)(x) = mr^{-1} f(x)$ for $(m-1)r^{-1} < x \leqslant mr^{-1}$. Then q_r is isomorphic to the location observable considered above and $q_r \to q$. Indeed, suppose $f_r \in H_r$ and $f_r \to f$. We can assume f is bounded and $|f_r| \leqslant M$. Then

$$\begin{aligned} \|q_r f_r - qf\| &\leqslant \|q_r f_r - qf_r\| + \|qf_r - qf\| \\ &\leqslant \|q_r f_r - qf_r\| + \|q\|\,\|f_r - f\| \end{aligned}$$

Then the last term approaches 0 as $r \to \infty$ and

$$\begin{aligned} \|q_r f_r - qf_r\|^2 &= \int_0^1 |q_r f_r - qf_r|^2\,dx \\ &= \sum_{m=1}^{r} \int_{(m-1)r^{-1}}^{mr^{-1}} |mr^{-1} f_r(x) - x f_r(x)|^2\,dx \\ &\leqslant M \sum_{m=1}^{r} \int_{(m-1)r^{-1}}^{mr^{-1}} |mr^{-1} - x|^2\,dx \leqslant M \sum_{m=1}^{r} r^{-3} \leqslant Mr^{-1} \to 0 \qquad \text{as } r \to \infty \end{aligned}$$

Now define p_r: $H_r \to H_r$ by $p_r = F^* q_r F$. Then for $f_n = \chi_{[(n-1)r^{-1}, nr^{-1}]}$ we have $e^{i(2\pi s/r)p_r} f_n = f_{n+s} \pmod r$, $s = 1, 2, \ldots, r$. Hence, for any $f \in H_r$ we have

$$\begin{aligned} \left[e^{i(2\pi s/r)p_r} f\right](x) &= \left[e^{i(2\pi s/r)p_r} \sum c_n f_n\right](x) = \sum c_n f_{n+s}(x) \\ &= \sum c_n f_n(x - sr^{-1}) = f(x - sr^{-1}) \end{aligned}$$

Thus, if $f_r \to f$, we have $e^{i(2\pi s_r/r)P_r} f_r \to e^{i2\pi sP} f$, where $s_r/r \to s[0,1]$. The above considerations can be extended to $L^2[0,\infty]$, and by redefining F they can also be extended to $L^2(\mathbb{R})$. Furthermore, we can extend these results to $L^2(\mathbb{R}^n)$.

We close this section by deriving explicit formulas for the matrix elements of P which will be convenient for later examples. Let $Q=\Sigma\, jQ_j$ be a location observable, and $P=\Sigma\, jP_j$ the corresponding motion observable. By differentiating the equation

$$\sum_{n=1}^{r} x^n = (x^{r+1}-x)(x-1)^{-1}$$

we obtain

$$\sum_{n=1}^{r} nx^n = \left[rx^{r+2}-(r+1)x^{r+1}+x\right](x-1)^{-2}$$

Hence, by some algebraic manipulation we obtain

$$P_{jk} = r^{-1}\sum_{n=1}^{r} ne^{2\pi in(k-j)/r} = \frac{1}{2}\left[1-\frac{i\cot\pi(k-j)}{r}\right]$$

for $j \neq k$ and

$$P_{jj} = r^{-1}\sum_{n=1}^{r} n = (r+1)/2$$

This gives the following formula for the matrix elements of the commutator;

$$[Q,P]_{jk} = \begin{cases} \dfrac{j-k}{2}\left[1+\dfrac{i\cot\pi(k-j)}{r}\right], & j\neq k \\ 0, & j=k \end{cases}$$

3. DYNAMICS

In this section we consider dynamics on $V=\mathbb{C}^r$. A vector $a=(a_1,\ldots,a_r)\in\mathbb{R}^r$ specifies a location observable $a\cdot\mathbf{Q}=\Sigma\, a_n Q_n$ and a motion observable $a\cdot\mathbf{P}=\Sigma\, a_n P_n$. We call $a\in\mathbb{R}^r$ the **basic location vector** for the system. The basic location vector also specifies a **free Hamiltonian** or free **energy observable** $H_0 = 2^{-1}(a\cdot\mathbf{P})^2 = 2^{-1}\Sigma\, a_n^2 P_n$. The matrix elements of H_0

become

$$(H_0)_{jk}=2^{-1}\sum_{n=1}^{r} a_n^2(P_n)_{jk}=\frac{1}{2r}\sum_{n=1}^{r} a_n^2 e^{2\pi i(k-j)/r}$$

For example, let $a_n=n-1$, $n=1,\ldots,r$. This would represent a particle located at one of the points $0,1,\ldots,r-1$. Then using the equation

$$\sum_{n=1}^{r}(n-1)^2x^n=\frac{r^2x^{r+1}}{x-1}-\frac{(2rx^2+1)x^2}{(x-1)^2}+\frac{x^{r+2}(x+1)}{(x-1)^3}$$

we obtain

$$(H_0)jk=\frac{1}{2r}\sum_{n=1}^{r}(n-1)^2e^{2\pi i(k-j)/r}$$

$$=\frac{1}{2}\left[r-\frac{1}{2}+\frac{1}{2}\cot^2\frac{\pi(k-j)}{r}+i\cot\frac{\pi(k-j)}{r}\right]$$

if $j\neq k$, and

$$(H_0)_{jj}=\frac{1}{2r}\sum_{n=1}^{r}(n-1)^2=\frac{(r-1)(2r-1)}{12}$$

Let us now return to a general basic location vector $a\in\mathbb{R}^r$. The dynamics is given by the one-parameter unitary group

$$U(t)=e^{-itH_0}=\sum_{n=1}^{r}e^{-ita_n^2/2}P_n,\qquad t\in\mathbb{R}$$

If the system is initially in the state e_k (the particle is located at the point $k-1$), then the probability that the system will be in the state e_j (the particle is located at the point $j-1$) at time t is

$$\mathcal{P}_{kj}(t)=|\langle e^{-itH_0}e_k,e_j\rangle|^2=|(e^{-itH_0})_{jk}|^2.$$

Now

$$(e^{-itH_0})_{jk}=\sum_{n=1}^{r}e^{-ita_n^2/2}(P_n)_{jk}=r^{-1}\sum_{n=1}^{r}\exp i\left[-ta_n^2/2+2\pi n(k-j)/r\right]$$

Hence,

$$|(e^{-itH_0})_{jk}|^2 = r^{-2} \sum_{n,m=1}^{r} \exp i\left[\frac{t}{2}(a_m^2 - a_n^2) + \frac{2\pi}{r}(n-m)(k-j)\right]$$

$$= r^{-2}\left\{r + 2\sum_{m>n=1}^{r} \cos\left[\frac{t}{2}(a_m^2 - a_n^2) + \frac{2\pi}{r}(n-m)(k-j)\right]\right\}$$

In particular, we have

$$|(e^{-itH_0})_{jj}|^2 = r^{-2}\left[r + 2\sum_{m>n=1}^{r} \cos\frac{t}{2}(a_m^2 - a_n^2)\right]$$

Important information about the system is given by the average location at time t given the initial state e_k. This is given by the following formula:

$$\langle Q\rangle_k(t) = \langle QU(t)e_k, U(t)e_k\rangle = \left\langle \sum_{j=1}^{r} a_j Q_j U(t)e_k, U(t)e_k \right\rangle$$

$$= \sum_{j=1}^{r} a_j \left\langle \langle U(t)e_k, e_j\rangle e_j, U(t)e_k \right\rangle$$

$$= \sum_{j=1}^{r} a_j \langle U(t)e_k, e_j\rangle\langle e_j, U(t)e_k\rangle$$

$$= \sum_{j=1}^{r} a_j |\langle U(t)e_k, e_j\rangle|^2 = \sum_{j=1}^{r} a_j |U(t)_{jk}|^2$$

The matrix elements of $|U(t)_{jk}|^2 = |(e^{-itH_0})_{jk}|^2$ can now be read off from the previous formulas.

So far we have worked with location states since they have important physical significance. However, we can also consider other pure states. Let $f \in V$ represent an arbitrary pure state. In the motion representation, we have

$$e^{-itH_0}f = e^{-itH_0} \sum_{n=1}^{r} \langle f, f_n\rangle f_n = \sum_{n=1}^{r} \langle f, f_n\rangle e^{-ita_n^2/2} f_n$$

If $g \in V$ represents another pure state, the transition probability at time t

becomes

$$|\langle e^{itH_0}f, g\rangle|^2 = \left|\left\langle \sum_{n=1}^{r} \langle f, f_n\rangle e^{ita_n^2/2} f_n, \sum_{m=1}^{r} \langle f, f_m\rangle f_m \right\rangle\right|^2$$

$$= \left| \sum_{n,m=1}^{r} \langle f, f_n\rangle\langle f_m, g\rangle e^{-ita_n^2/2}\, \delta_{nm}\right|^2$$

$$= \left| \sum_{n=1}^{r} \langle f, f_n\rangle\langle f_n, g\rangle e^{-ita_n^2/2}\right|^2$$

The transition probability at time t in the location representation is

$$|\langle e^{-itH_0}f, g\rangle|^2 = \left|\left\langle e^{-itH_0} \sum_{j=1}^{r} f(j)e_j, \sum_{k=1}^{r} g(k)e_k \right\rangle\right|^2$$

$$= \left| \sum_{j,k=1}^{r} f(j)\overline{g(k)}\,(e^{itH_0})_{jk}\right|^2$$

Now suppose the system finds itself in a potential $V = b\cdot\mathbf{Q}$, $b \in \mathbb{R}^r$, and the free energy has the general form $H_0 = 2^{-1}(a\cdot\mathbf{P})^2$. The total energy observable becomes

$$H = H_0 + V = 2^{-1}(a\cdot\mathbf{P})^2 + b\cdot\mathbf{Q} = 2^{-1}\sum a_n^2 P_n + \sum b_n Q_n$$

The matrix elements of H are

$$H_{jk} = \frac{1}{2r}\sum_{n=1}^{r} a_n^2 e^{2\pi i(k-j)/r} + b_j\delta_{jk}$$

In particular, the average energy in the state e_j becomes

$$\langle H\rangle_j = \langle He_j, e_j\rangle = H_{jj} = \frac{1}{2r}\sum_{n=1}^{r} a_n^2 + b_j$$

In general, the eigenvalues and eigenvectors of H are extremely difficult to calculate and an explicit formula for the dynamics e^{-itH} cannot be given. In the next section we shall give some specific low-dimension examples that can be solved exactly. If the eigenvalues λ_n and corresponding eigenvectors

ψ_n, $n=1,\ldots,r$, are known, then the matrix elements of $U(t)=e^{-itH}$ become

$$U(t)_{jk}=\langle e^{-itH}e_k, e_j\rangle=\sum_{n=1}^{r}\langle e_k,\psi_n\rangle\psi_n, e_j\rangle e^{-it\lambda_n}$$

Knowing these we can then compute the transition probability

$$\mathcal{P}_{kj}(t)=|U(t)_{jk}|^2 \quad \text{and} \quad \langle Q\rangle_k(t)=\sum_{j=1}^{r}a_j|U(t)_{jk}|^2$$

Although the eigenvalues and eigenvectors of $H=H_0+V$ cannot be found exactly in general, approximations can be found using perturbation theory. Let $H=H_0+\varepsilon V$, where $H_0=2^{-1}(a\cdot\mathbf{P})^2$ and $V=b\cdot\mathbf{Q}$. Then for ε sufficiently small, the eigenvalues $\lambda_n(\varepsilon)$ and corresponding eigenvectors $f_n(\varepsilon)$ of H are analytic (Kato, 1966) and we have

$$\lambda_n(\varepsilon)=\lambda_n+\varepsilon\lambda_{n,1}+\varepsilon^2\lambda_{n,2}+\cdots$$

$$f_n(\varepsilon)=f_n+\varepsilon f_{n,1}+\varepsilon^2 f_{n,2}+\cdots$$

for $n=1,\ldots,r$. Clearly, $\lambda_n(0)=\lambda_n=a_n^2/2$, the nth eigenvalue of H_0 and $f_n(0)=f_n=F^*e_n$, the corresponding eigenvector. Since

$$(H_0+\varepsilon V)(f_n+\varepsilon f_{n,1}+\varepsilon^2 f_{n,2}+\cdots)=(\lambda_n+\varepsilon\lambda_{n,1}+\cdots)(f_n+\varepsilon f_{n,1}+\cdots)$$

comparing coefficients of ε gives

$$H_0 f_{n,1}+Vf_n=\lambda_n f_{n,1}+\lambda_{n,1}f_n$$

Taking inner products with f_n gives

$$\langle f_{n,1}, H_0 f_n\rangle+\langle Vf_n, f_n\rangle=\lambda_n\langle f_{n,1}, f_n\rangle+\lambda_{n,1}$$

Since $H_0 f_n=\lambda_n f_n$ we have

$$\lambda_{n,1}=\langle Vf_n, f_n\rangle=r^{-1}\sum b_j=r^{-1}\operatorname{tr}V$$

Taking inner products with f_m, $m\neq n$, gives

$$\lambda_m\langle f_{n,1}, f_m\rangle+\langle Vf_n, f_m\rangle=\lambda_n\langle f_{n,1}, f_m\rangle$$

Hence,

$$\langle f_{n,1}, f_m\rangle = (\lambda_n - \lambda_m)^{-1}\langle Vf_n, f_m\rangle$$
$$= (\lambda_n - \lambda_m)^{-1} r^{-1} \sum_j b_j e^{2\pi i j(m-n)/r}$$

and

$$f_{n,1} = \sum_{m\neq n} \langle f_{n,1}, f_m\rangle f_m$$

is solved explicitly. Higher-order perturbation terms can be computed but they become more complicated.

4. EXAMPLES

Let $V = \mathbf{C}^2$ and let $a = (0,1) \in \mathbb{R}^2$ be the basic location vector. The location observable is $Q = a\cdot\mathbf{Q} = 0Q_1 + 1Q_2 = Q_2$. The finite Fourier transform is

$$F = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

and the first and second motion observables become

$$P_1\begin{bmatrix} 2^{-1} & -2^{-1} \\ -2^{-1} & 2^{-1} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 2^{-1} & 2^{-1} \\ 2^{-1} & 2^{-1} \end{bmatrix}$$

The motion observable is $P = a\cdot\mathbf{P} = 0P_1 + 1P_2 = P_2$ and the free energy observable is $H_0 = 2^{-1}P^2 = 2^{-1}(a\cdot\mathbf{P})^2 = 2^{-1}P_2$. The dynamical group now has the form

$$U(t) = e^{-itH_0} = e^{-it0}P_1 + e^{-it/2}P_2 = P_1 + e^{-it/2}P_2$$
$$= \frac{1}{2}\begin{bmatrix} 1+e^{-it/2} & -1+e^{-it/2} \\ -1+e^{-it/2} & 1+e^{-it/2} \end{bmatrix}$$

The transition probability $\mathcal{P}_{kj}(t)$ from the location state e_k to the location state e_j after time t becomes

$$\mathcal{P}_{kj}(t) = |U(t)_{jk}|^2 = \tfrac{1}{2}\left\{1 + \cos\left[\tfrac{1}{2}t + \pi(k-j)\right]\right\}.$$

In particular, $\mathcal{P}_{jj}(t)=\frac{1}{2}(1+\cos\frac{1}{2}t)$. Thus, if a particle is initially at location 0, then it will oscillate back and forth between 0 and 1 with period 4π. This becomes particularly clear by noting that

$$\langle Q\rangle_1(t)=\sum_{j=1}^{2} a_j|U(t)_{jk}|^2=\frac{1}{4}|e^{-it/2}-1|^2=\frac{1}{2}\left(1-\cos\frac{t}{2}\right).$$

Now suppose the system is in a potential $V=b\cdot\mathbf{Q}$, $b=(b_1,b_2)\in\mathbb{R}^2$. The total energy becomes

$$H=H_0+V=2^{-1}P_2+b_1Q_1+b_2Q_2=\frac{1}{4}\begin{bmatrix}1+b_1 & 1\\ 1 & 1+b_2\end{bmatrix}.$$

The eigenvalues of H are

$$\lambda_1=8^{-1}\left[2+b_1+b_2-\left(4+(b_1-b_2)^2\right)^{1/2}\right]$$

$$\lambda_2=8^{-1}\left[2+b_1+b_2+\left(f+(b_1-b_2)^2\right)^{1/2}\right]$$

Letting $c=b_2-b_1$, the corresponding normalized eigenvectors are

$$\psi_1=N_1\left(1,2^{-1}\left[c-(4+c^2)^{1/2}\right]\right)$$

$$\psi_2=N_2\left(1,2^{-1}\left[c+(4+c^2)^{1/2}\right]\right)$$

where N_1, N_2 are the normalization constants

$$N_1=2^{1/2}\left\{4+c\left[c-(4+c^2)\right]^{1/2}\right\}^{-1/2}$$

$$N_2=2^{1/2}\left\{4+c\left[c+(4+c^2)\right]^{1/2}\right\}^{-1/2}$$

The matrix elements of $U(t)=e^{-itH}$ are

$$U(t)_{jk}=\langle e_k,\psi_1\rangle\langle\psi_1,e_j\rangle e^{-it\lambda_1}+\langle e_k,\psi_2\rangle\langle\psi_2,e_j\rangle e^{-it\lambda_2}$$

Now suppose the system represents a particle whose admissible locations are 0 and 1. If the particle is initially at the point 0, then the probability the particle will be at the point 0 at time t is

$$\mathcal{P}_{11}(t)=|U(t)_{11}|^2=|N_1^2e^{-1t\lambda_1}+N_2^2e^{-it\lambda_2}|^2$$
$$=N_1^4+N_2^4+2N_1^2N_2^2\cos(\lambda_2-\lambda_1)t$$
$$=[4+c^2]^{-1}\left[2+c^2+2\cos 4^{-1}(4+c^2)^{1/2}t\right]$$

Notice that

$$[4+c^2]^{-1}c^2\leqslant\mathcal{P}_{11}(t)\leqslant 1$$

and $\mathcal{P}_{11}(t)$ oscillates between these two values. As $|b_2-b_1|\to 0$ then $\mathcal{P}_{11}(t)$ approaches the free case and as $|b_2-b_1|\to\infty$, $\mathcal{P}_{11}(t)$ approaches 1 for all t. Thus, as the potential difference gets large, the particle tends to remain at its initial position. If the particle is initially at 0, the average position of the particle is

$$\langle Q\rangle_1(t)=|U(t)_{21}|^2=1-|U(t)_{11}|^2=1-\mathcal{P}_{11}(t)$$

It follows that on the average, the particle remains between 0 and $4[4+c^2]^{-1}$. Again, as the potential difference $|b_2-b_1|$ gets large, the particle remains near its initial position on the average.

The physical significance of the energy eigenstates ψ_1, ψ_2 in terms of the particle position may be seen as follows. The average position in the state ψ_1 is

$$\langle Q\rangle_{\psi_1}=\langle Q\psi_1,\psi_1\rangle=\langle Q_2\psi_1,\psi_1\rangle=|\langle\psi_1,e_2\rangle|^2$$
$$=\tfrac{1}{4}N_1^2\left[c-(4+c^2)^{1/2}\right]^2$$
$$=\left[2+c^2-c(4+c^2)^{1/2}\right]\left[4+c^2-c(4+c^2)^{1/2}\right]^{-1}$$

Similarly,

$$\langle Q\rangle_{\psi_2}=\left[2+c^2+c(4+c^2)^{1/2}\right]\left[4+c^2+c(4+c^2)^{1/2}\right]^{-1}$$

For $c=0$, we get the free case $\langle Q\rangle_{\psi_1}=\langle Q\rangle_{\psi_2}=\frac{1}{2}$. For $c\neq 0$, we have $\langle Q\rangle_{\psi_1}<\langle Q\rangle_{\psi_2}$. Thus, in the lower-energy state ψ_1, the particle is closer to 0 than in the higher-energy state ψ_2. Also, notice that $\langle Q\rangle_{\psi_1}+\langle Q\rangle_{\psi_2}=1$ and $\langle Q\rangle_{\psi_1}\to 0$, $\langle Q\rangle_{\psi_2}\to 1$ as $c\to\infty$.

For our next example, we consider $V=\mathbb{C}^4$ and let $a=(-3,-1,3,1)\in\mathbb{R}^4$ be the basic location vector. We shall apply formulas in Sections 2 and 3 to compute various observables and quantities. The location observable $Q=a\cdot\mathbf{Q}=\Sigma a_n Q_n$ becomes

$$Q=\begin{bmatrix}-3 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

The j-motion observables, $j=1,2,3,4$ are

$$P_1=\frac{1}{4}\begin{bmatrix}1 & i & -1 & -i\\ -i & 1 & i & -1\\ -1 & -i & 1 & i\\ i & -1 & -i & 1\end{bmatrix},\qquad P_2=\frac{1}{4}\begin{bmatrix}1 & -1 & 1 & -1\\ -1 & 1 & -1 & 1\\ 1 & -1 & 1 & -1\\ -1 & 1 & -1 & 1\end{bmatrix}$$

$$P_3=\frac{1}{4}\begin{bmatrix}1 & -i & -1 & i\\ i & 1 & -i & -1\\ -1 & i & 1 & -i\\ -i & -1 & i & 1\end{bmatrix},\qquad P_4=\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}$$

The free energy observable $H_0=\frac{1}{2}9P_1+\frac{1}{2}P_2+\frac{1}{2}9P_3+\frac{1}{2}P_4$ is

$$H_0=\frac{1}{2}\begin{bmatrix}5 & 0 & -4 & 0\\ 0 & 5 & 0 & -4\\ -4 & 0 & 5 & 0\\ 0 & -4 & 0 & 5\end{bmatrix}$$

The free dynamical group $U(t)=e^{-itH_0}=e^{-it9/2}P_1+e^{-it/2}P_2+e^{-it9/2}P_3+e^{-it/2}P_4$ becomes

$$U(t)=\frac{1}{2}\begin{bmatrix}e^{-it9/2}+e^{-it/2} & 0 & e^{-it9/2}+e^{-it/2} & 0\\ 0 & e^{-it9/2}+e^{-it/2} & 0 & -e^{-it9/2}+e^{-it/2}\\ -e^{-i9/2}+e^{-it/2} & 0 & e^{-it9/2}+e^{-it/2} & 0\\ 0 & -e^{-it9/2}+e^{-it/2} & 0 & e^{-it9/2}+e^{-it/2}\end{bmatrix}$$

The transition probabilities become

$$\mathcal{P}_{jj}(t)=2^{-1}(1+\cos 4\pi t), \qquad \mathcal{P}_{jj+1}(t)=\mathcal{P}_{j+1j}(t)=0$$

$$\mathcal{P}_{jj+2}(t)=\mathcal{P}_{j+2,j}(t)=2^{-1}(1-\cos 4\pi t)$$

The average location is $\langle Q\rangle_k(t)=a_k\cos 4\pi t$.

Now suppose the system experiences a potential $V=b\cdot\mathbf{Q}$, $b=2^{-1}(b_1,b_2,b_3,b_4)\in\mathbb{R}^4$. The total energy is $H=H_0+V$. The energy eigenvalues become

$$\lambda_1=\frac{10+b_1+b_3+\left[(b_1-b_3)^2+64\right]^{1/2}}{4}$$

$$\lambda_2=\frac{10+b_1+b_3-\left[(b_1-b_3)^2+64\right]^{1/2}}{4}$$

$$\lambda_3=\frac{10+b_2+b_4+\left[(b_2-b_4)^2+64\right]^{1/2}}{4}$$

$$\lambda_4=\frac{10+b_2+b_4-\left[(b_2-b_4)^2+64\right]^{1/2}}{4}$$

and the corresponding (unnormalized) eigenvectors are

$$\phi_1=\begin{bmatrix}1\\0\\\dfrac{5+b_1-\lambda_1}{4}\\0\end{bmatrix},\qquad \phi_2=\begin{bmatrix}1\\0\\\dfrac{5+b_1-\lambda_2}{4}\\0\end{bmatrix}$$

$$\phi_3=\begin{bmatrix}0\\1\\0\\\dfrac{5+b_2-\lambda_3}{4}\end{bmatrix},\qquad \phi_4=\begin{bmatrix}0\\1\\0\\\dfrac{5+b_2-\lambda_4}{4}\end{bmatrix}$$

One can now compute $\mathcal{P}_{jk}(t)$ and $\langle Q\rangle_k(t)$ using previous formulas.

The reason that everything was easy to compute in the last example was because of the symmetry of the basic location vector $a\in\mathbb{R}^4$. This resulted in degeneracies in H_0 and the system could be described by two uncoupled two-dimensional parts. Suppose now we have an unsymmetrical basic location vector such as $a=(0,1,2,3)\in\mathbb{R}^4$. Then the situation is much more complicated. For example,

$$P=\frac{1}{4}\begin{bmatrix} 6 & 2-2i & 2 & 2+2i \\ 2+2i & 6 & 2-2i & 2 \\ 2 & 2+2i & 6 & 2-2i \\ 2-2i & 2 & 2+2i & 6 \end{bmatrix}$$

and

$$U(t)=P_1+e^{-it/2}P_2+e^{-i2t}P_3+e^{-it9/2}P_4$$

The diagonal transition probabilities become

$$\mathcal{P}_{jj}(t)=\frac{1}{16}\left[4+2\left(\cos\frac{t}{2}+\cos 2t+\cos\frac{9}{2}t+\cos\frac{3}{2}t+\cos 4t+\cos\frac{5}{2}t\right)\right]$$

The particle now experiences a complicated periodic motion.

5. DISCRETE QUANTUM MECHANICS

We have shown earlier that as the dimension approaches infinity, the finite-dimensional quantum mechanics approaches the usual quantum mechanics. However, we could also jump immediately to an infinite-dimensional discrete quantum mechanics. Let $I=\{0,\pm 1,\pm 2,\ldots\}$ be the set of integers and let V be the Hilbert space $l^2(I)=\{f\colon I\to\mathbb{C}:\Sigma|f(n)|^2<\infty\}$. The natural location observable is the operator Q on V with domain $D(Q)=\{f\in V\colon nf(n)\in V\}$ defined by $(Qf)(n)=nf(n)$ for all $f\in D(Q)$. Since we have no Fourier transform on V, it is not clear how the motion observable P should be defined. However, it is reasonable to assume from our previous work that $(e^{imP}f)(n)=f(n-m)$, n, $m\in I$.

It turns out to be more convenient to work in the Hilbert space $L^2[0,2\pi]$ using the unitary transformation T: $V\to L^2[0,2\pi]$ given by

$$(Tf)(x)=(2\pi)^{-1/2}\sum f(n)e^{inx}$$

The transformed location observable $\hat{Q}$ becomes

$$\begin{aligned}(\hat{Q}f)(x)&=(TQT^{-1}f)(x)=\left[T\left(n\left\langle f,(2\pi)^{-1/2}e^{inx}\right\rangle\right)\right](x)\\ &=(2\pi)^{-1}\sum n\langle f,e^{inx}\rangle e^{inx}=-i\frac{d}{dx}(2\pi)^{-1}\sum\left\langle f,e^{inx}\right\rangle e^{inx}\\ &=-i\frac{d}{dx}f(x)\end{aligned}$$

To be precise, $D(\hat{Q})$ is the set of absolutely continuous functions f on $[0,2\pi]$ such that $f'\in L^2[0,2\pi]$ and $f(0)=f(2\pi)$. For $f\in D(\hat{Q})$, $\hat{Q}f=-if'$. The transformed motion observable $\hat{P}$ satisfies

$$\left(e^{im\hat{P}}f\right)(x)=\left(Te^{imP}T^{-1}f\right)(x)=\left(T\langle f,(2\pi)^{-1/2}e^{i(n-m)x}\rangle\right)(x)$$
$$=(2\pi)^{-1}\sum\left\langle f,e^{i(n-m)x}\right\rangle e^{inx}=e^{imx}f(x)$$

We can therefore define $(\hat{P}f)(x)=xf(x)$ for all $f\in L^2[0,2\pi]$.

We now drop the caret on Q and P and work exclusively on $L^2[0,2\pi]$. The location eigenvectors are the orthonormal basis $\psi_j=(2\pi)^{-1}e^{ijx}$. The free energy observable is $H_0=2^{-1}P^2=2^{-1}x^2$ and the free dynamical group is $U(t)=e^{-ix^2t/2}$. The transition probability from the jth to kth location state is

$$\mathcal{P}_{jk}(t)=(2\pi)^{-1}\left|\int\exp\left\{-i\left[x(2^{-1}t)^{1/2}+(k-j)(2t)^{-1/2}\right]^2\right\}dx\right|^2$$

For $t\neq 0$ this can be written as

$$\mathcal{P}_{jk}(t)=(2\pi)^{-1}\left|\int\exp\left\{-i\left[x(2^{-1}t)^{1/2}+(k-j)(2t)^{-1/2}\right]^2\right\}dx\right|^2$$

This gives a Fresnel integral whose values may be obtained from tables.

The average location given the initial state ψ_j is

$$\begin{aligned}\langle Q\rangle_j(t)&=(2\pi)^{-1}\langle QU(t)\psi_j,\psi_j\rangle\\&=(2\pi)^{-1}\left\langle -i\frac{d}{dx}e^{-ix^2t/2+ijx},e^{-ix^2t/2+ijx}\right\rangle\\&=(2\pi)^{-1}\int_0^{2\pi}(-xt+j)\,dx=-t\pi+j\end{aligned}$$

This represents a particle moving at constant velocity $-\pi$.

Suppose we have a potential $-aQ_j$, $a>0$, in location space. This corresponds to the operator $V=-aQ_j$ on $L^2[0,2\pi]$, where Q_j is the projection onto the vector ψ_j. The total energy observable is $H=x^2/2-aQ_j$. Solving the eigenvalue equation $H\psi=E\psi$ gives

$$\psi=a\langle\psi,\psi_j\rangle(x^2/2-E)^{-1}\psi_j$$

Taking the inner product with ψ_j gives

$$a^{-1}=\pi^{-1}\int_0^{2\pi}(x^2-2E)^{-1}dx$$

It follows that $E<0$ and that E is the solution of the equation $a^{-1}=\pi^{-1}(-2E)^{-1/2}\tan^{-1}2\pi(-2E)^{-1/2}$. Thus H has the single eigenvalue E and the rest of its spectrum is continuous.

As another example, suppose we have a potential $V=Q$. Then $H=x^2/4-i\,d/dx$ and the eigenvalue problem $H\psi=E\psi$ has solutions $\psi=ce^{-i(Ex-x^3/3)}$. To satisfy the periodic condition for $D(Q)$ we must have $\psi(0)=\psi(2\pi)$. This implies that E must have the form $E_n=n+(2\pi)^2/3$, $n=0,\pm1,\pm2,\ldots$. The corresponding normalized eigenvectors are $\phi_n=(2\pi)^{-1}e^{i(E_nx-x^3/3)}$. The average location in the energy eigenstate ϕ_n is

$$\langle Q\rangle_n=\langle Q\phi_n,\phi_n\rangle=(2\pi)^{-1}\int_0^{2\pi}(E_n-x^2)\,dx=n$$

6. SYMMETRIC HILBERT SPACE

In this section and the following section we shall describe finite-dimensional nonrelativistic quantum field theories. In particular, we shall consider the symmetric Hilbert space for $V=\mathbb{C}^r$ in the present section and the antisymmetric Hilbert space for V in the next section.

Let $V=\mathbb{C}^r$ and $SV=\mathbb{C}\oplus V\oplus(V\circledS V)\oplus(V\circledS V\circledS V)\oplus\cdots$ be the symmetric Hilbert space for V, where $\circledS$ denotes the symmetric tensor product. Now SV is unitarily equivalent to $L^2(\mathbb{R}^r)$ in a natural way. Indeed, let $e_1,\ldots,e_r$ be the standard basis for $\mathbb{C}^r$ and let $h_n\in L^2(\mathbb{R})$ be the Hermite function

$$h_n(x)=(2^nn!)^{-1/2}(-1)^n\pi^{-1/4}e^{x^2/2}\left(\frac{d}{dx}\right)^ne^{-x^2}$$

$n=0,1,\ldots$. Then h_n is a orthonormal basis for $L^2(\mathbb{R})$ and

$$h_{j_1,\ldots,j_r}(x_1,\ldots,x_r)=h_{j_1}(x_1)\cdots h_{j_r}(x_r)$$

$j_1,\ldots,j_r=0,1,\ldots,$ is an orthonormal basis for $L^2(\mathbb{R}^r)$. On the orthonormal basis $e_1^{j_1}\circledS\cdots\circledS e_r^{j_r}$ define

$$J_e\left(e_1^{j_1}\circledS\cdots\circledS e_r^{j_r}\right)=h_{j_1\cdots j_r}(x_1,\ldots,x_r)$$

$j_1, \ldots, j_r = 0, 1, \ldots$. Extending J_e by linearity and closure gives a unitary operator $J_e\colon SV \to L^2(\mathbb{R}^r)$. In this way, $L^2(\mathbb{R}^r)$ may be thought of as the symmetric Hilbert space for V. In particular, the pure states for the usual nonrelativistic quantum mechanics on $L^2(\mathbb{R}^3)$ can be written as superpositions of states for three-dimensional "particles" in $\mathbb{C}^3$.

Define the **annihilation operators** $A(e_j)$ on $L^2(\mathbb{R}^r)$ by $A(e_j) = 2^{-1/2}(x_j + \partial/\partial x_j)$, $j = 1, \ldots, r$. Their adjoints are the **creation operators** $A^*(e_j) = 2^{-1/2}(x_j - \partial/\partial x_j)$, $j = 1, \ldots, r$. These operators satisfy

$$A(e_k)h_{j_1 \cdots j_r} = j_k^{1/2} h_{j_1 \cdots j_k - 1 \cdots j_r}$$

$$A^*(e_k)h_{j_1 \cdots j_r} = (j_k + 1)^{1/2} h_{j_1 \cdots j_k + 1 \cdots j_r}$$

and the commutation relations $[A(e_j), A(e_k)] = [A^*(e_j), A^*(e_k)] = 0, [A(e_j), A^*(e_k)] = \delta_{jk}$.

For arbitrary $f \in \mathbb{C}^r$, we define $A(f) = \Sigma \langle e_j, f \rangle A(e_j)$ and $A^*(f) = A(f)^* = \Sigma \langle f, e_j \rangle A^*(e_j)$. These operators satisfy

$$A(\alpha f + \beta g) = \bar{\alpha} A(f) + \bar{\beta} A(g)$$

$$A^*(\alpha f + \beta g) = \alpha A^*(f) + \beta A^*(g)$$

and the commutation relations $[A(f), A(g)] = [A^*(f), A^*(g)] = 0$, $[A(f), A^*(g)] = \langle g, f \rangle$ for all $f, g \in \mathbb{C}^r$, $\alpha, \beta \in \mathbb{C}$. In terms of differential operators we can write

$$A(f) = 2^{-1/2} \sum \bar{f}_j \left(x_j + \frac{\partial}{\partial x_j} \right) = 2^{-1/2} (\bar{\mathbf{f}} \cdot \mathbf{x} + \bar{\mathbf{f}} \cdot \nabla)$$

$$A^*(f) = 2^{-1/2} \sum f_j \left(x_j - \frac{\partial}{\partial x_j} \right) = 2^{-1/2} (\mathbf{f} \cdot \mathbf{x} - \mathbf{f} \cdot \nabla)$$

We also define the self-adjoint field operators $\pi(f), \psi(f)$ by

$$\pi(f) = 2^{-1/2}[A(f) + A^*(f)] = \mathrm{Re}\,\mathbf{f} \cdot \mathbf{x} - i\,\mathrm{Im}\,\mathbf{f} \cdot \nabla$$

$$\psi(f) = -i 2^{-1/2}[A(f) - A^*(f)] = -\mathrm{Im}\,\mathbf{f} \cdot \mathbf{x} - i\,\mathrm{Re}\,\mathbf{f} \cdot \nabla$$

For $f \in \mathbb{C}^r$ a real vector, we have $\pi(f) = \mathbf{f} \cdot \mathbf{x}$ and $\psi(f) = -i\mathbf{f} \cdot \nabla$. In particular, $\pi(e_j) = x_j$ and $\psi(e_j) = -i\partial/\partial x_j$, the usual position and momentum operators. Notice that for f restricted to $\mathbb{R}^r$, $f \to (\pi(f), \psi(f))$ gives a quantum field on $L^2(\mathbb{R}^r)$ in the following sense. The operators $\pi(f), \psi(f)$,

$f \in \mathbb{R}^r$, are essentially self-adjoint on a common dense, invariant domain with cyclic vector $h_{0\cdots 0}$. The maps $f \to \pi(f)$, $f \to \psi(f)$ are linear and continuous in the strong operator topology. Moreover, π and ψ are universally convariant. That is, if G is any group of unitary operators on $\mathbb{R}^r$, then there exists a unitary representation $g \to U_g$ of G on $L^2(\mathbb{R}^r)$ such that $U_g h_{0\cdots 0} = h_{0\cdots 0}$ and $U_g \pi(f) U_g^{-1} = \pi(gf)$, $U_g \psi(f) U_g^{-1} = \psi(gf)$, for all $g \in G$, $f \in \mathbb{R}^r$. Indeed, define U_g on $L^2(\mathbb{R}^r)$ by $(U_g h)(\mathbf{x}) = h(g^{-1}\mathbf{x})$. It is easy to check that $g \to U_g$ is a unitary representation. Also,

$$\begin{aligned}(U_g h_{0\cdots 0})(\mathbf{x}) &= h_{0\cdots 0}(g^{-1}\mathbf{x}) = \pi^{-r/4} e^{-\langle g^{-1}\mathbf{x},\, g^{-1}\mathbf{x}\rangle/2} \\ &= \pi^{-r/4} e^{-\langle \mathbf{x},\mathbf{x}\rangle/2} = h_{0\cdots 0}(\mathbf{x})\end{aligned}$$

and

$$\begin{aligned}\left[U_g \pi(f) U_g^{-1} h\right](\mathbf{x}) &= \left[U_g \pi(f) U_{g^{-1}} h\right](\mathbf{x}) = \left[\pi(f) U_g h\right](g^{-1}\mathbf{x}) \\ &= \mathbf{f} \cdot g^{-1}\mathbf{x}\, U_g h(g^{-1}\mathbf{x}) = \mathbf{f} \cdot g^{-1}\mathbf{x}\, h(\mathbf{x}) = g\mathbf{f} \cdot \mathbf{x}\, h(\mathbf{x}) \\ &= \left[\pi(gf) h\right](\mathbf{x})\end{aligned}$$

We also define the j**th number operators**

$$N(e_j) = A^*(e_j) A(e_j) = 2^{-1}\left(-\frac{\partial^2}{\partial x_j^2} + x_j^2 - I\right)$$

$j = 1, \ldots, r$. These operators satisfy

$$N(e_k) h_{j_1 \cdots j_r} = j_k h_{j_1 \cdots j_r}$$

so the eigenvectors of $N(e_k)$ are the basis vectors $h_{j_1 \cdots j_r}$ and the eigenvalues are the nonnegative integers.

The **total number operator** is $N(e) = \Sigma N(e_k)$ and satisfies

$$N(e) h_{j_1 \cdots j_r} = \left(\sum k_j\right) h_{j_1 \cdots j_r}$$

The eigenvectors of $N(e)$ are the basis vectors $h_{j_1 \cdots j_2}$ and the eigenvalues are the nonnegative integers. All of these are multiple eigenvalues except 0. The differential operator for $N(e)$ is

$$N(e) = 2^{-1}\left(-\sum \frac{\partial^2}{\partial x_j^2} + \sum x_j^2 - rI\right)$$

We now find the eigenvectors of the annihilation operators $A(e_j)$. Suppose $g\in L^2(\mathbb{R}^r)$ and $A(e_1)g=\lambda g$, $\lambda\in\mathbb{C}$. Suppose g has the form $g(x_1,\ldots,x_r)=u(x_1)v(x_2,\ldots,x_r)$. Then

$$\frac{du}{dx_1}=(2^{-1/2}-x_1)u(x_1)$$

Solving this differential equation gives

$$g(x_1)=ce^{2^{1/2}\lambda x_1}e^{-x_1^2/2}$$

and hence

$$g(x_1,\ldots,x_r)=ce^{2^{1/2}\lambda x_1}e^{-x_1^2/2}v(x_2,\ldots,x_r).$$

Thus $A(e_1)$ has a dense set of eigenvectors in $L^2(\mathbb{R}^r)$ and every $\lambda\in\mathbb{C}$ is an eigenvalue. Hence, a set of simultaneous eigenvectors for $A(e_1),\ldots,A(e_r)$ with eigenvalues $\lambda_1,\ldots,\lambda_r\in\mathbb{C}$, respectively, is

$$\pi^{-r/4}\exp\left(-\tfrac{1}{2}\sum\lambda_j^2\right)\exp\left(\sum 2^{1/2}\lambda_j x_j\right)\exp\left(-\sum x_j^2/2\right)$$

We denote this function by $\exp(\Sigma\lambda_j e_j)$, and if $f=\Sigma\lambda_j e_j$, simply by $\exp f$. The vector $\exp f$ represents the pure state in which there are an infinite number of "elementary" systems all in the state $f\in\mathbb{C}^r$. In fact, it is not hard to show that

$$J_e^{-1}(\exp f)=\left(1,f,\frac{f\otimes f}{(2!)^{1/2}},\frac{f\otimes f\otimes f}{(3!)^{1/2}},\ldots\right)$$

A simple calculation shows that

$$\exp f=\exp\left(\tfrac{1}{2}\sum\langle f,e_j\rangle^2\right)h_{0\cdots 0}\left(x_1-2^{1/2}\langle f,e_1\rangle,\ldots,x_r-2^{1/2}\langle f,e_r\rangle\right)$$

In particular, $\exp 0=h_{0\cdots 0}$, and $\exp e_j=e^{1/2}h_{0\cdots 0}(x_1,\ldots,x_j-2^{1/2},\ldots,x_r)$.

If one tries to solve the eigenvalue equation $A^*(e_1)g=\lambda g$, $\lambda\in\mathbb{C}$, for the creation operator $A^*(e_1)$, then the only solutions are of the form

$$g(x_1,\ldots,x_r)=ce^{-2^{1/2}x_1}e^{x_1^2/2}v(x_2,\ldots,x_r)$$

Since $g\notin L^2(\mathbb{R}^r)$, the creation operators have no eigenvalues.

It is interesting to note that we have shown that $L^2(\mathbb{R}^r)$ is the second quantization of $\mathbb{C}^r$. The finite-dimensional Hilbert space

$$[L^2(\mathbb{R}^r)]_n=\mathrm{sp}\{h_{j_1\cdots j_r}: j_1+\cdots+j_r=n\}$$

represents the space of n-particle states and

$$L^2(\mathbb{R}^r) = \bigoplus_{n=0}^{\infty} \left[L^2(\mathbb{R}^r)\right]_n$$

The operators $A^*(e_k)A(e_l)$ leave the subspaces $[L^2(\mathbb{R}^r)]_n$ invariant and

$$A^*(e_k)A(e_l)h_{j_1\cdots j_r} = (j_k+1)^{1/2} j_l^{1/2} h_{j_1\cdots j_k+1\cdots j_l-1\cdots j_r}$$

for $k \neq l$ $[A^*(e_k)A(e_k) = N(e_k)]$.

We now consider second quantization of operators. Let $H: V \to V$ be a normal linear operator with normalized eigenvectors $h_1, \ldots, h_r$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_r$. The **second quantization** of H is the operator $\Gamma(H)$ on $L^2(\mathbb{R}^r)$ defined by $\Gamma(H) = \Sigma \lambda_n A^*(h_n)A(h_n)$. In terms of the location eigenvectors $e_1, \ldots, e_r$ this becomes

$$\Gamma(H) = \sum_{n,j,k} \lambda_n \langle h_n, e_j \rangle \langle e_k, h_n \rangle A^*(e_j)A(e_k)$$

The (unbounded) normal operator $\Gamma(H)$ has a complete set of eigenvectors $J_e J_h^{-1} h_{j_1\cdots j_r}$ and corresponding eigenvalues $\Sigma \lambda_n j_n$, $j_n \in \{0, 1, 2, \ldots\}$. In particular, we see that $\Gamma(Q_n) = A^*(e_n)A(e_n) = N(e_n)$ and $\Gamma(I) = \Sigma A^*(e_n)A(e_n) = N(e)$. Moreover,

$$\Gamma(P_n) = \frac{1}{r} \sum_{j,k} e^{2\pi i n(k-j)/r} A^*(e_j)A(e_k)$$

Now let $U(t) = e^{-itH}$ be a dynamical group on V. The corresponding dynamical group on $L^2(\mathbb{R}^r)$ is defined as $\hat{U}(t) = e^{-it\Gamma(H)}$. It is well known that $\hat{U}(t)\exp f = \exp U(t)f$, for all $f \in \mathbb{C}^r$ (Guichardet, 1972; Klauder, 1970).

We now compute some expectations and transition probabilities. First of all

$$\begin{aligned}\Gamma(H)h_{j_1\cdots j_r} &= \sum_{n,m,p} \lambda_n \langle h_n, e_m \rangle \langle e_p, h_n \rangle A^*(e_m)A(e_p) h_{j_1\cdots j_r} \\ &= \sum_{\substack{n,m,p \\ m\neq p}} \lambda_n \langle h_n, e_m \rangle \langle e_p, h_n \rangle (j_m+1)^{1/2} j_p^{1/2} h_{j_1\cdots j_m+1\cdots j_p-1\cdots j_r} \\ &\quad + \sum_{n,m} \lambda_n |\langle h_n, e_m \rangle|^2 j_m h_{j_1\cdots j_r}\end{aligned}$$

Hence,

$$\langle \Gamma(H)h_{j_1\cdots j_r}, h_{j_1'\cdots j_r'}\rangle = 0$$

unless $j_1' = j_1, \ldots, j_r' = j_r$ or $j_m' = j_m + 1$ and $j_p' = j_p - 1$ and $j_k' = j_k$ for all $k \neq m, p$, and some m, p. In the first case we have the expectation of $\Gamma(H)$ in the state $h_{j_1\cdots j_r}$,

$$\langle \Gamma(H)h_{j_1\cdots j_r}, h_{j_1\cdots j_r}\rangle = \sum_{n,m} \lambda_n j_m |\langle h_n, e_m\rangle|^2$$

and in the second case we have

$$\langle \Gamma(H)h_{j_1\cdots j_r}, h_{j_1'\cdots j_r'}\rangle = \sum_n \lambda_n \langle h_n, e_m\rangle\langle e_p, h_n\rangle (j_m + 1)^{1/2} j_p^{1/2}$$

If $f \in L^2(\mathbb{R}^r)$ is an arbitrary unit vector in the domain of $\Gamma(H)$, then the expectation of $\Gamma(H)$ in the state f is

$$\langle \Gamma(H)f, f\rangle = \sum_{\substack{j_1,\ldots,j_r\\ j_1',\ldots,j_r'}} \langle f, h_{j_1\cdots j_r}\rangle\langle h_{j_1'\cdots j_r'}, f\rangle \langle \Gamma(H)h_{j_1\cdots j_r}, h_{j_1'\cdots j_r'}\rangle$$

and the expression can be computed using our previous formulas. In particular, in the nr-particle state $h_{n\cdots n}$ we have

$$\langle \Gamma(H)h_{n\cdots n}, h_{n\cdots n}\rangle = n\sum_{k,m} \lambda_k |\langle h_k, e_m\rangle|^2 = n\sum_k \lambda_k$$

In the n-particle state $h_{0\cdots n\cdots 0}$ (n in the jth place) we have

$$\langle \Gamma(H)h_{0\cdots n\cdots 0}, h_{0\cdots n\cdots 0}\rangle = n\sum_k \lambda_k |\langle h_k, e_j\rangle|^2$$

To find the expectation of $\Gamma(H)$ in the state $e^{-1/2}\exp e_1$ notice that

$$\exp e_1 = \sum_{n=0}^{\infty} (n!)^{-1/2} h_{n0\cdots 0}$$

and hence

$$\begin{aligned}\langle \Gamma(H)\exp e_1, \exp e_1\rangle &= \sum_n (n!)^{-1}\langle \Gamma(H)h_{n\cdots 0}, h_{n\cdots 0}\rangle\\ &= \sum_n (n!)^{-1} n\sum_k \lambda_k |\langle h_k, e_1\rangle|^2 = e\sum_k \lambda_k |\langle h_k, e_1\rangle|^2.\end{aligned}$$

To find the transition probability $|\langle e^{-it\Gamma(H)}f, g\rangle|^2$, let $g_{j_1\cdots j_r} = J_e J_h^{-1} h_{j_1\cdots j_r}$ be the eigenvectors of $\Gamma(H)$ with corresponding eigenvalues $\Sigma\lambda_n j_n$. Then

$$\left\langle e^{-it\Gamma(H)}f, g\right\rangle = \sum_{j_1,\ldots,j_r} \langle f, g_{j_1\cdots j_r}\rangle\langle g_{j_1\cdots j_r}, g\rangle e^{-it\Sigma\lambda_n j_n}$$

Suppose that $h_j = \Sigma_k c_{jk}e_k$. If $n = \Sigma j_k$, then

$$g_{j_1\cdots j_r} = J_e\left(h_1^{\otimes j_1} ⓢ \cdots ⓢ h_r^{\otimes j_r}\right)$$
$$= \left[\frac{n!}{j_1!\cdots j_r!}\right]^{1/2} c_{11}^{j_1}c_{21}^{j_2}\cdots c_{r1}^{j_r}h_{n0\cdots 0} + \cdots$$

Then, for example, if $\Sigma j_k = n$, we have the expectation

$$\left\langle e^{-it\Gamma(H)}h_{n0\cdots 0}, h_{n0\cdots 0}\right\rangle = \sum_{j_1,\ldots,j_r} n!(j_1!\cdots j_r!)^{-1}|c_{11}^{j_1}\cdots c_{r1}^{j_r}|^2 e^{-it\Sigma\lambda_k j_k}$$
$$= \left[|c_{11}|^2 e^{-it\lambda_1} + \cdots + |c_{r1}|_e^{2-it\lambda_r}\right]^n$$
$$= \left(\Sigma|\langle h_j, e_1\rangle|^2 e^{-it\lambda_j}\right)^n = \langle e^{-itH}e_1, e_1\rangle^n$$

Moreover,

$$\left\langle e^{-it\Gamma(H)}\exp e_1, \exp e_1\right\rangle = \sum_{j_1,\ldots,j_r} (j_1!\cdots j_r!)^{-1}|c_{11}^{j_1}\cdots c_{r1}^{j_r}|^2 e^{-it\Sigma\lambda_k j_k}$$
$$= \exp\left(\Sigma|c_{k1}|^2 e^{-it\lambda_k}\right) = \exp\langle e^{-itH}e_1, e_1\rangle$$

This can also be seen from

$$\left\langle \hat{U}(t)\exp e_1, \exp e_1\right\rangle = \left\langle \exp U(t)e_1, \exp e_1\right\rangle = \exp\left\langle U(t)e_1, e_1\right\rangle$$

In general,

$$\left\langle U(t)\exp f, \exp g\right\rangle = \exp\left\langle U(t)f, g\right\rangle$$

7. ANTISYMMETRIC HILBERT SPACE

As before $V = \mathbb{C}^r$ with standard basis $e_1,\ldots,e_r$. The antisymmetric Hilbert space for V is defined by

$$AV = \mathbb{C} \oplus V \oplus \left(V ⓐ V\right) \oplus \cdots \oplus \left(V ⓐ V ⓐ \cdots ⓐ V\right)$$

where V appears r times in the last term. The dimension of $V^{ⓐn}$ is $\binom{r}{n}=r!/n!(n-r)!$ so the dimension of AV is $\Sigma_{n=1}^{r}\binom{r}{n}=2^r$. We now define a unitary transformation $J_e\colon AV\to\mathbb{C}^{2^r}$. Let $e_0, e_1,\ldots, e_{2^r-1}$ be the standard basis for $\mathbb{C}^{2^r}$. For $n\in\{0,1,\ldots,2^r-1\}$ write n in its binary notation $n=j_1j_2\cdots j_r$, $j_k=0$ or 1, and let $e_n=e_{j_1\cdots j_r}$. For example, in $\mathbb{C}^2$, $e_0=e_{00}$, $e_1=e_{01}$, $e_2=e_{10}$, $e_3=e_{11}$. For the basis element $e_{i_1}ⓐ\cdots ⓐe_{i_k}$, $i_n\in\{1,\ldots,r\}$, $i_1<i_2<\cdots<i_k$, in AV, define

$$J_e(e_{i_1}ⓐ\cdots ⓐe_{i_k})=e_{j_1\cdots j_r}$$

where $j_n=1$ if $i_m=n$ for some n and $j_n=0$ otherwise. For example, in $\mathbb{C}^2$, $J_e(1)=e_{00}$, $J_e(e_1)=e_{10}$, $J_e(e_2)=e_{01}$, $J_e(e_1ⓐe_2)=e_{11}$. Then J_e, defined above, extends to a unitary transformation from AV onto $\mathbb{C}^{2^r}$.

We define the *annihilation operators* $A(e_j)$, $j=1,\ldots,r$, on $\mathbb{C}^{2^r}$ by $A(e_k)e_{j_1\cdots j_r}=0$ if $j_k=0$, and $A(e_k)e_{j_1\cdots j_r}=e_{j_1\cdots j_k-1\cdots j_r}$ if $j_k=1$. Their adjoints are the *creation operators*, $A^*(e_k)e_{j_1\cdots j_r}=0$ if $j_k=1$, and $A^*(e_k)e_{j_1\cdots j_r}=e_{j_1\cdots j_k+1\cdots j_r}$ if $j_k=0$. For arbitrary $f\in\mathbb{C}^r$ we define $A(f)=\Sigma\langle e_j,f\rangle A(e_j)$ and $A^*(f)=A(f)^*=\Sigma\langle f,e_j\rangle A^*(e_j)$. These satisfy the usual anticommutation relations. The operator $A(e_k)$ $[A(e_k)^*]$ has the single eigenvalue 0 with corresponding eigenvectors $e_{j_1\cdots j_r}$, where $j_k=0$ $(j_k=1)$, $j_n=0$ or 1, $n\neq k$.

As before, we define the self-adjoint *field operators* $\pi(f)=2^{-1/2}[A(f)+A^*(f)]$, $\psi(f)=-2^{-1/2}i[A(f)-A^*(f)]$. We also define the jth *number operators* $N(e_j)=A^*(e_j)A(e_j)$ and the *total number operator* $N(e)=\Sigma N(e_j)$. Then

$$N(e_k)e_{j_1\cdots j_r}=j_ke_{j_1\cdots j_r}$$

and

$$N(e)e_{j_1\cdots j_r}=(\Sigma k_j)e_{j_1\cdots j_r}$$

As an example, if $r=1$, we have $V=\mathbb{C}$ and $AV\cong\mathbb{C}^2$. Then $A(1)e_0=0$, $A(1)e_1=e_0$, $A^*(1)e_0=e_1$, $A^*(1)e_1=0$. Hence,

$$A(1)=\begin{bmatrix}0&1\\0&0\end{bmatrix},\qquad A^*(1)=\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

The field operators become

$$\pi(1)=2^{-1/2}[A(1)+A^*(1)]=2^{-1/2}\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

$$\psi(1)=-2^{-1/2}i[A(1)-A^*(1)]=2^{-1/2}\begin{bmatrix}0&-i\\i&0\end{bmatrix}$$

These together with the location observable $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ give essentially, the Pauli spin matrices.

As in the symmetric case, we define the *second quantization* of a normal operator H on V by $\Gamma(H) = \Sigma\lambda_n A^*(h_n)A(h_n)$. Many of the results of the previous section hold for $\Gamma(H)$ except those involving the exponential vectors, for which there seems to be no counterpart in the antisymmetric case. In fact, the expressions are simpler in the antisymmetric case since $j_k = 0$ or 1.

This gives a hierarchy of second quantizations. For example, $\mathbb{C}^{16} = A^3\mathbb{C}$, and $L^2(\mathbb{R}^4) = SA^2\mathbb{C}$.

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